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APPROXIMATE METHODS FOR THE TREATMENT OF DIFFERENTIAL EQUATIONS.

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Various approximate rules of integration devised for the purpose of dealing with functions which do not admit of expression in analytical form are well known. It is the purpose of the present paper to call attention to various allied methods for the approximate treatment of such differential equations as do not admit of treatment by the usual analytical methods. For purposes of illustration only two of the familiar rules for integration will be used.

Let $y_0 y_1$, etc., denote consecutive values of the function separated by equal increments Δx. Then by the trapezoidal rule the integral of the function for one interval is :—

$$A_0' = \frac{\Delta x}{2}(y_0 + y_1),$$

while by the so-called Simpson's one-third, or Stirling's rule, the integral for two intervals is :—

$$A_0^2 = \frac{\Delta x}{3}(y_0 + 4y_1 + y_2).$$

If for convenience we denote successive derivatives with reference to x by $y'\ y''\ y'''$, etc., we shall have from the trapezoidal rule :—

$$y_1 = y_0 + \frac{\Delta x}{2}(y_0' + y_1') \tag{1}$$

$$y_1' = y_0' + \frac{\Delta x}{2}(y_0'' + y_1''), \tag{2}$$

or by substituting from (2) in (1)

$$y_1 = y_0 + \Delta x y_0' + \frac{\overline{\Delta x}^2}{4}(y_0'' + y_1''). \tag{3}$$

Similarly for Simpson's rule we have :—

$$y_2 = y_1 + \frac{\Delta x}{3}(y_0' + 4y_1' + y_2') \tag{4}$$

$$y_2' = y_0' + \frac{\Delta x}{3}(y_0'' + 4y_1'' + y_2''), \tag{5}$$

or by substitution from (5) in (4)

$$y_2 = y_0 + \frac{\Delta x}{3}(2y_0' + 4y_1') + \frac{\overline{\Delta x}^2}{9}(y_0'' + 4y_1'' + y_2''). \tag{6}$$

From Maclaurin's theorem we may also deduce similar expressions which may be used in like manner, though they are relatively somewhat less accurate than those derived above. For an increment Δx we thus have :—

$$y_1 = y_0 + y_0'\Delta x + \tfrac{1}{2}\, y_0''\overline{\Delta x}^2 + \tfrac{1}{6}\, y_0'''\overline{\Delta x}^3 + \text{\&c.,} \tag{7}$$

$$y_1' = y_0' + y_0''\Delta x + \tfrac{1}{2}\, y_0'''\overline{\Delta x}^2 + \text{\&c.,} \tag{8}$$

$$y_1'' = y_0'' + y_0'''\Delta x + \text{\&c.} \tag{9}$$

Now taking any differential equation, of the second order for example, suppose that we have given the particular values at a starting point which we may denote by subscript 0. Denoting an adjacent point by subscript 1, let us substitute in the equation the values of y_1' and y_1 as given in (2) and (3). The resulting equation will then express y_1'' as a function of y_0 y_0' and y_0'' and will be of degree equal to that of the term of highest degree involving y or its derivatives. Hence if the original equation is linear the resulting value of y_1'' is readily found and this substituted in (2) and (1) will give values of y_1' and y_1 which will necessarily satisfy the differential equation at this point. The point (1) being thus determined, it is in turn considered as known and the next adjacent point is determined in a similar manner. In this way by a step by step process the values of y are found for a series of points covering the range for which the history of the function is desired. If the equation is linear it is usually preferable to solve analytically for y_1'' and then substitute in this solution the known values, thus determining the successive values of y_1'' and thence those of y_1' and y_1. If the equation is not linear, special means suitable to the case may be adopted to effect the solution. In such cases, particularly where transcendental functions are involved, it may be necessary to treat the equation by tentative and approximate methods rather than by direct solution. To this end a value may be assumed for y_1'' and from (1) (2) and (3) the resultant approximate values of y_1' and y_1 found. These values of y_1' and y_1 being substituted in the differential equation, the latter may then be solved for y_1'' by such means as may be appropriate or necessary. A comparison of this result with the assumed value will furnish a basis for the correction of the latter, and thus by a few trials a set of values may be found which will satisfy all conditions.

It will be noted that the essential feature of the process is the substitution of an algebraic equation for the original differential equation, and the solution of the latter is thus made to depend upon that of the former.

If two values of the original function are known, one by approximate determination as above for example, then the values of y_2' and y_2 as given in

(5) and (6) may be substituted and the equation solved for y_2''. This being substituted back will give values of y_2' and y_2, and thus the next point becomes known. In like manner the values for points (1) and (2) may be used to determine point (3), and so forward over the range for which the function is desired.

The values of y_1 y_1' y_1'', etc., given in (7) (8) and (9) may if desired be used is the same general manner as those above considered.

The errors involved in these operations will be those due to the departure of the function treated by the rules for approximate integration, from forms for which such rules are accurate. Such errors may be reduced to any desired amount by taking the values of Δx sufficiently small. Some increase in accuracy may also be gained by the use of rules of higher orders. In general, however, it will be found preferable in all approximate integrations to limit the rules to those of the first and second orders as given above, and to control the question of resultant error by an appropriate selection of interval Δx. The trapezoidal or rule of the first order assumes between points 0–1, 1–2, etc., a linear law, and will therefore require a relatively close spacing of ordinates.

The Simpson's rule or rule of the second order assumes between points 0–2, 1–3, etc., a second degree parabolic law. With spacing the same as for the trapezoidal rule the accuracy will be considerably greater, or the same accuracy may be attained with a wider spacing or greater value of Δx. By remembering the nature of the error for the trapezoidal rule, and noting whether the functions y'' and y' are increasing or decreasing, convex or concave to x, the tendency of the errors due to this rule may be readily ascertained.

If certain quantities in the original differential equation are given graphically as by a curve for which no analytical equation is known, the methods above outlined are applicable without change, and one of the chief fields of usefulness for such methods will be found in the treatment of equations of this character. Further reference to such application will be found in the examples given below.

As with the treatment of differential equations in general, special manipulation or transformation may be useful in particular cases, but no general rules can be given for such cases, and the broad outlines as given above will serve to indicate the general line of treatment when the solution is to be effected by these methods.

The general application of the methods herein presented is evidently independent of the order of the equation, and except for the greater amount of work involved in carrying out the operations, those of the third or fourth or of higher orders are quite as readily treated as those of the first or second.

These methods may also be sometimes used to advantage with simultaneous differential equations when the attempt at elimination would result in an equation too complicated for treatment in the usual manner. An illustration is found in the examples given below.

Differential equations involving imaginaries or complex quantities are as readily treated by these methods as are any other. That is, the existence of complex or imaginary quantities gives rise to no additional difficulty of treatment beyond the added work required to carry out the operations involved.

Except in particular cases these methods do not seem likely to be of much aid in the treatment of partial differential equations. The quantities are here so related that usually a general solution must first be found, and then made particular by a substitution of boundary or other conditions, and this is not possible by a step by step process such as that here described.

The work involved in the determination of an unknown from a differential equation by the methods above outlined will depend wholly on circumstances. In many cases it might be prohibitive, in others not. It will depend on the nature of the functions involved, on the accuracy desired, and on the range to be covered between the known and the unknown. The question of the availability of such methods will then depend on these facts, and on the time available and justifiable for the determination of the value desired. The same considerations affect the final accuracy which may be made whatever is desired by the means already pointed out. The present paper is not concerned with the question of the length or shortness of the operations involved, but rather with the development of means whereby results may be obtained if their work will justify the labor necessary.

Illustrations.

As an example made up simply to show the application of the method to an equation too complicated for solution by analytical means, take the following:—

$$\left[y\frac{dy}{dx}\frac{d^2y}{dx^2}\right] = \sin\left[x\frac{dy}{dx}\right] + \log\left[y\frac{dy}{dx}\right]$$

with the initial conditions $x = 0$, $y = 1$, $\frac{dy}{dx} = 1$, $\frac{d^2y}{dx^2} = 0$. This equation has been treated by the above method with the trapezoidal rule, taking $\Delta x = .2$, between the limits $x = 0$ and $x = 3$. The graphical histories of y and its two derivatives are given in Fig. 1.

To show the error in a given case the method was applied to a Bessel's equation of zero order, and the value of $J_0(x)$ was found from $x = 0$ to $x = 1$ by the trapezoidal rule and with intervals of .1. The resultant value of y_1'' in terms of known quantities is in this case readily seen to be :—

$$y_1'' = -\frac{\left[y_0' + \frac{\Delta x}{2} y_0''\right] + x\left[y_0 + \Delta x y_0' + \frac{\overline{\Delta x}^2}{4} y_0''\right]}{\left[x + \frac{\Delta x}{2} + x\frac{\overline{\Delta x}^2}{4}\right]}.$$

In the following table the first column gives the values thus found, and the second, the true values as found from the series in the usual way :—

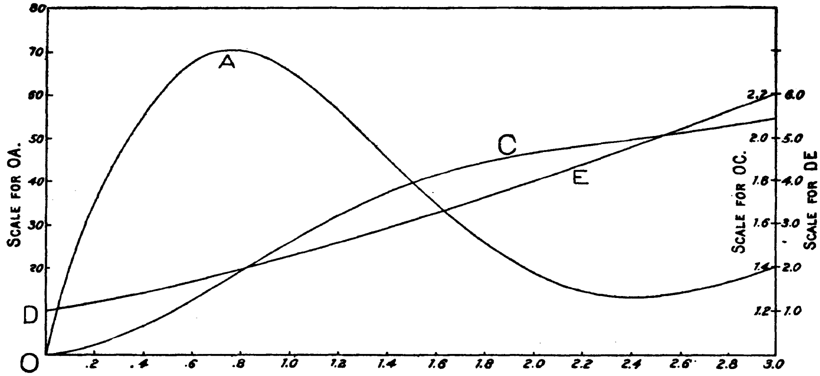
x	$J_0(x)$	
0	1.0000	1.0000
.1	.9975	.9975
.2	.9903	.9903
.3	.9776	.9776
.4	.9604	.9604
.5	.9385	.9385
.6	.9121	.9120
.7	.8813	.8812
.8	.8464	.8463
.9	.8077	.8075
1.0	.7654	.7652

In the case of the common pendulum giving rise to the equation :—

$$\frac{d^2\theta}{dt^2} + \frac{g}{e}\sin\theta = 0$$

a well known elliptic integral furnishes the solution. The evaluation of this integral fails however when the initial displacement is very large on account of the slow convergence of the series. Thus let the initial displacement be 3.1406 or .001 less than π. The series in such case is of no avail and other means must be sought. By the methods above described the period of oscillation for a seconds pendulum is found in such case to be 5.725 sec.

FIG. 1.

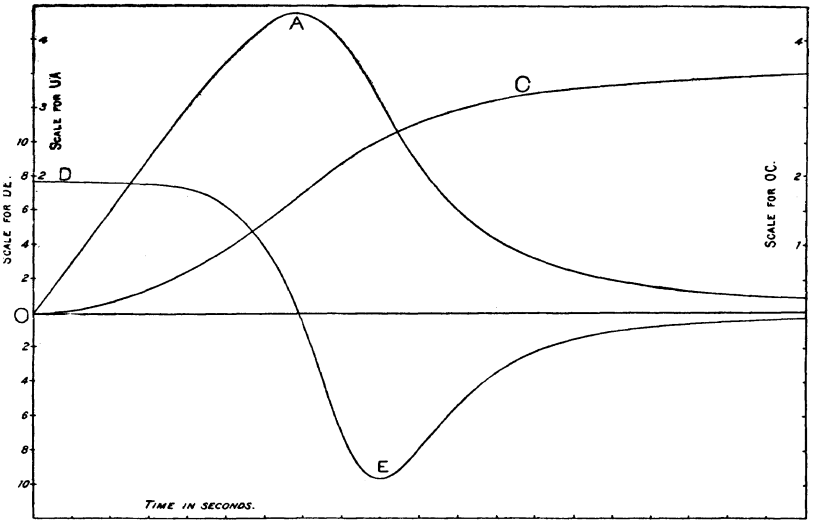


OA gives history of $\frac{d^2y}{dx^2}$

OC " " " $\frac{dy}{dx}$

DE " " " y

FIG. 2.



DE gives history of $\frac{d^2\theta}{dt^2}$

OA " " " $\frac{d\theta}{dt}$

OC " " " θ

As a test of accuracy the period for an initial displacement of 30° was computed by the same means and compared with the result given by the series. The first gave 1.017 sec. and the second 1.0174. This was by the trapezoidal rule and $\Delta t = .1$ sec.

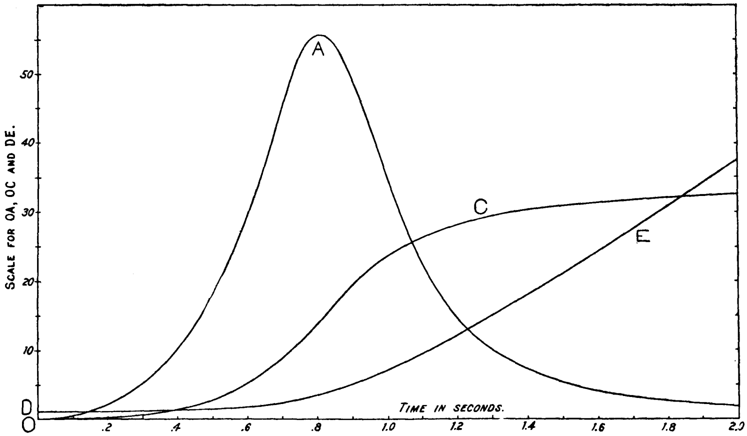
Given the following problem. A vertical rod is provided with a horizontal arm indefinite in extent, on which may slide without friction a weight m located initially at a distance a from the axis of the rod. Let the vertical rod be urged to rotate about its own axis by a constant torque such as would be produced by a falling weight W acting by means of a cord led over a fixed pulley to a drum attached to the rod. It is required to find the motion of the weight m. This gives rise to the following simultaneous equations :—

$$\frac{d^2r}{dt^2} = r\left[\frac{d\theta}{dt}\right]^2$$

$$gW = a(W + B)\frac{d^2\theta}{dt^2} + \frac{mr}{a}\left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right].$$

Their solution by analytical means seems very difficult. By the approximate means above outlined, however, the problem may be solved and the history of the various functions derived. These are illustrated in Figs. 2 and 3 for the following numerical values: $W = 10$, $B = 10$, $a = 2$, $r_1 = 1$, $m = 4$.

FIG: 3.



OA gives history of $\frac{d^2r}{dt^2}$

OC " " " $\frac{dr}{dt}$

DE " " " r

In the case of an electric circuit without iron acted on by an alternating electro-motive force, the general equation is :—

$$f(t) = Ri + L \frac{\partial i}{\partial t}$$

where $f(t)$ represents the history of the electro-motive force and gives simply the successive ordinates of a geometrical curve, the characteristics of which may be found experimentally, but of which no equation is known. This represents an important class of problems in which a portion of the data is graphical and not algebraical. The solution by the method here presented is simple and very satisfactory, and from its special interest to engineering students has been given by the author elsewhere.*

Illustrative examples might be widely multiplied, but enough has been given to show the application of the methods herein developed, and the possibility of obtaining satisfactory practical results in cases where the equations are beyond the reach of the usual analytical methods.

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* Sibley Journal of Engineering, Feb., 1897.